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A NOTE ON TESTS FOR MONOTONE FAILURE RATE BASED ON INCOMPLETE DATA

by

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ABSTRACT

Tests for exponential versus IFRA distributions based on incomplete data are defined and shown to be unbiased. The tests are motivated by a class of tests considered in detail by Bickel and Doksum. Tests for exponential versus IFR distributions based on the ranks of total time on test statistics are also considered.

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Richard E. Barlow[†] and Frank Proschan

1. INTRODUCTION AND SUMMARY

Let $0 \equiv X_{(0)} \leq X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the order statistics of a (complete) random sample from a population with distribution F and density f such that $F(0) = 0$. Bickel and Doksum (1968) consider the problem of testing

$$H_0 : F(t) = 1 - e^{-\lambda t} \qquad t \geq 0 , \lambda > 0$$

versus

$$H_1 : F \text{ IFR}$$

(i.e., $-\log[1 - F(t)]$ convex on $[0,\infty)$).

Let $D_i = (n-i+1)(X_{(i)} - X_{(i-1)})$, $i = 1,2, \dots, n$. They consider tests based on statistics of the form

$$\frac{\sum_{i=1}^{n} a_i D_i}{\sum_{i=1}^{n} D_i}$$

where $a_1 \geq a_2 \geq \dots \geq a_n$. The test, ϕ_a, rejects H_0 when $\sum_{i=1}^{n} a_i D_i \Big/ \sum_{i=1}^{n} D_i \geq c_{\alpha,a,n}$. They compute the asymptotic relative efficiency of

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various such tests relative to selected parametric alternatives. Such tests were shown to be unbiased against IFRA (for increasing failure rate average) alternatives by Barlow and Proschan (1966) and hence a fortiori for IFR alternatives. [See also Birnbaum, Esary and Marshall (1965) for justification of the IFRA assumption.]

The purpose of this note is to show that analogous tests designed to treat incomplete samples of failure data are also unbiased against IFRA alternatives. Let X_i be the time to failure of the i^{th} item in a sample of size n. Let L_i be a given truncation time for the i^{th} item and let

$$Z_i = \min(X_i, L_i) \qquad i = 1, 2, \ldots, n .$$

Let $0 \equiv Z_{(0)} \leq Z_{(1)} \leq \cdots \leq Z_{(k)}$ be the first k observed failure times. Note that "withdrawals" may occur between $Z_{(i)}$ and $Z_{(i+1)}$ and that k is, in general, a random variable. Let $n(u)$ be the (random) number of items on test at time u.

We define a test, ϕ_a^*, (a modification of ϕ_a) which rejects H_0 in favor of

$$H_1' : F \text{ IFRA}$$

(i.e., $-\{\log[1 - F(t)]\}/t$ nondecreasing on $[0,\infty)$) when

$$W_a = \frac{\sum_{i=1}^{k} a_i \int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du}{\int_0^{Z_{(k)}} n(u)du} \geq c^*_{\alpha,a,k} .$$

Note that $\int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du$ represents the total time on test between the $i-1^{st}$ and

i^{th} observed failures. The distribution of W_a can be computed under H_0 using the fact that $Y_i = \int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du$ $(i = 1,2, \ldots, k)$ are distributed as independent exponential random variables under H_0 conditioned on the value of k. We show that ϕ_a^* is an unbiased test for IFRA alternatives for weights $a = (a_1, a_2, \ldots, a_n)$ for which $a_1 \geq a_2 \geq \ldots \geq a_n$.

2. DISTRIBUTION OF W_a UNDER H_0

Let $r(t) = f(t)/[1 - F(t)]$ be the failure rate function for F We will need the following lemma, stated without proof in Bray, Crawford, and Proschan (1967).

Lemma 1.

For any distribution F $(F(0) = 0)$ with failure rate $r(t)$,

$Y_i = \int_{Z_{(i-1)}}^{Z_{(i)}} r(u)n(u)du$, $i = 1,2, \ldots, k$ are independently distributed with density e^{-y}.

Proof:

Let $Y_1 = \int_0^{Z_{(1)}} r(u)n(u)du$ and $S_0(t) = \int_0^t r(u)n(u)du$. Note that $S_0(t)$ is well defined up to the time of the first observed failure since $n(u)$ depends only on the specified truncation times L_i $(i = 1,2, \ldots, n)$ up until $Z_{(1)}$. Then

$$P[Y_1 > y_1] = P[S_0(Z_{(1)}) > y_1] = P\left[Z_{(1)} > S_0^{-1}(y_1)\right] =$$

$$= \exp\left[-S_0\left(S_0^{-1}(y_1)\right)\right] = e^{-y_1}.$$

Thus Y_1 has density e^{-y_1}.

Now let $Y_2 = \int_{Z_{(1)}}^{Z_{(2)}} r(u)n(u)du$ and $S_{x_1}(t) = \int_{x_1}^t r(u)n(u)du$. Note that conditionally on $Z_{(1)} = x_1$, S_{x_i} is well defined for $x_1 \leq t < Z_{(2)}$. Hence

$$P[Y_2 > y_2 \mid Z_{(1)} = x_1] = P\left[S_{x_1}(Z_{(2)}) > y_2 \mid Z_{(1)} = x_1\right] =$$

$$= P\left[Z_{(2)} > S_{x_1}^{-1}(y_2) \mid Z_{(1)} = x_1\right] = \exp\left[-S_{x_1}\left(S_{x_1}^{-1}(y_2)\right)\right] = e^{-y_2} .$$

Thus Y_2 is independent of Y_1 and also exponentially distributed with mean 1. If we continue in this manner, conditioning on previous events, we establish the lemma. ||

Under H_0 , $r(t) \equiv \lambda$ and we see from the lemma that, given k observed failures,

$$W_a \overset{st}{=} \frac{\sum_{i=1}^{k} a_i Y_i}{\sum_{j=1}^{k} Y_j} ,$$

where $\overset{st}{=}$ denotes stochastic equality and $Y_1, Y_2, \ldots, Y_k$ are independent, exponentially distributed random variables with unit mean.

3. UNBIASEDNESS UNDER IFRA ALTERNATIVES

We need the following lemma to establish unbiasedness. Define

$$R(t) = \int_0^t r(u)du \quad \text{and} \quad T(t) = \int_0^t n(u)du .$$

Lemma 2.

If $\frac{R(t)}{t}$ is nondecreasing in $t \geq 0$, $n(t) \geq 0$, and $\frac{T(t)}{t}$ is nonincreasing in $t \geq 0$, then

$$\text{(i)} \quad r(t) \geq \frac{\int_0^t r(u)du}{t} \geq \frac{\int_0^t r(u)dT(u)}{T(t)}$$

$$\text{(ii)} \quad \frac{\int_0^t r(u)dT(u)}{T(t)} \text{ is nondecreasing in } t \geq 0 ,$$

when the indicated integrals exist.

Proof:

To show (i). The first inequality follows from differentiating $\frac{R(t)}{t}$. Since $\frac{R(t)}{t} \geq 0$ is nondecreasing in $t \geq 0$, we can approximate $R(t)$ arbitrarily closely from below by a positive linear combination of functions of the form

$$R(t) = \begin{cases} 0 & 0 \leq t < x \\ t & t \geq x \end{cases}$$

[cf. Barlow, Marshall, and Proschan (1967)]. By the Lebesque monotone convergence theorem, we need only establish the second inequality in (ii) for functions $R(t)$ of this type. Hence for $t \geq x$,

$$\frac{\int_0^t n(u)dR(u)}{T(t)} = \frac{n(x)x + \int_x^t n(u)du}{T(t)} = 1 + \frac{xn(x) - T(x)}{T(t)} .$$

This is nondecreasing in $t \geq x$ since (a) $T(t)$ is nondecreasing in $t \geq 0$, and (b) $xn(x) - T(x) \leq 0$ since $\frac{T(x)}{x}$ is nonincreasing in $x \geq 0$.

To show (ii). Clearly

$$\frac{d}{dt}\left[\frac{\int_0^t r(u)n(u)du}{\int_0^t n(u)du}\right] \geq 0$$

if and only if

$$r(t)n(t) \int_0^t n(u)du \geq n(t) \int_0^t r(u)n(u)du$$

which follows from (i). ||

Note that if $r(t)$ is nondecreasing in $t \geq 0$, then (ii) follows for all $n(t) \geq 0$.

Lemma 2 may be used in testing for IFRA in models other than the one described in the introduction; see for example the model of Bray, Crawford, and Proschan (1967).

Theorem 1.

If F is IFRA with failure rate $r(t)$ and $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(k)}$ are the observed failure times, $n(t) \geq 0$ for $t \geq 0$, and $\frac{T(t)}{t} \geq 0$ is nonincreasing in $t \geq 0$, then (conditional on k),

$$W_a = \frac{\sum_{i=1}^{k} a_i \int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du}{\int_0^{Z_{(k)}} n(u)du} \overset{st}{\geq} \frac{\sum_{i=1}^{k} a_i Y_i}{\sum_{i=1}^{k} Y_i}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $Y_1, Y_2, \ldots, Y_k$ are independently distributed as exponential random variables with unit mean.

Proof:

Since $n(u) \geq 0$ and $T(t)/t$ is nonincreasing, Lemma 2 applies, yielding

$$\frac{\beta_i}{\alpha_i} = \frac{\int_0^{Z_{(i)}} r(u)n(u)du}{\int_0^{Z_{(i)}} n(u)du}$$

nondecreasing in $i = 1, 2, \ldots, k$. By Lemma 1 we need only show that

$$(1) \qquad \frac{\sum_{i=1}^{k} a_i \int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du}{\int_0^{Z_{(k)}} n(u)du} \geq \frac{\sum_{i=1}^{k} a_i \int_{Z_{(i-1)}}^{Z_{(i)}} r(u)n(u)du}{\int_0^{Z_{(k)}} r(u)n(u)du}$$

i.e.,

$$\frac{\sum_{i=1}^{k} a_i(\alpha_i - \alpha_{i-1})}{\alpha_k} \geq \frac{\sum_{i=1}^{k} a_i(\beta_i - \beta_{i-1})}{\beta_k}$$

where $\alpha_0 = \beta_0 \equiv 0$. Note that

$$\sum_{i=1}^{k} a_i(\alpha_i - \alpha_{i-1}) = (a_1 - a_2)\alpha_1 + (a_2 - a_3)\alpha_2 + \ldots + a_k\alpha_k = \sum_{i=1}^{k} \Delta_i\alpha_i$$

where $\Delta_i = a_i - a_{i-1} \geq 0$ for $i = 1,2, \ldots, k-1$ and $\Delta_k = a_k$. Hence $\frac{\beta_i}{\alpha_i} \leq \frac{\beta_k}{\alpha_k}$ implies $\sum_{i=1}^{k} \frac{\Delta_i\alpha_i}{\alpha_k} \geq \sum_{i=1}^{k} \frac{\Delta_i\beta_i}{\beta_k}$, which proves (1). ||

4. APPLICATION OF TOTAL TIME ON TEST

Assuming an exponential distribution, the results of Bickel and Doksum (1968) may be used to establish the asymptotic normality of W_a in the incomplete data case for selected vectors $a = (a_1, \ldots, a_k)$. Perhaps, the most useful test is the total time on test statistic. In the case of a complete sample of size n , this is S_1^* in the Bickel-Doksum paper, obtained by choosing $a_i = -i/(n+1)$, after algebraic manipulation. In the case of incomplete data as described in the introduction, with k failures observed; the total time on test statistic is

$$W_{a^\circ} = \frac{\sum_{i=1}^{k-1} (k-i) \int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du}{\int_0^{Z_{(k)}} n(u)du} ,$$

obtained by choosing $a^\circ = (k-1, k-2, \ldots, 1, 0)$.

The exact distribution conditioned on the number of observed failures $k \geq 2$ is easily computed in this case. Table 1 is a short table of percentage points. Note that, under H_0

$$W_{a^\circ} \overset{st}{=} U_1 + U_2 + \cdots + U_{k-1}$$

when U_i $(i = 1,2, \ldots, k-1)$ are independent uniform random variables on $[0,1]$. Since the distribution of W_{a° is symmetric about $\frac{k-1}{2}$, we tabulate upper percentiles only.

TABLE 1: PERCENTILES χ_α OF TOTAL TIME ON TEST STATISTIC, W_{a°

k-1 \ α	.900	.950	.975	.990	.995
2	1.553	1.684	1.776	1.859	1.900
3	2.157	2.331	2.469	2.609	2.689
4	2.753	2.953	3.120	3.300	3.411
5	3.339	3.565	3.754	3.963	4.097
6	3.917	4.166	4.376	4.610	4.762
7	4.489	4.759	4.988	5.244	5.413
8	5.056	5.346	5.592	5.869	6.053
9	5.619	5.927	6.189	6.487	6.683
10	6.178	6.504	6.781	7.097	7.307
11	6.735	7.077	7.369	7.702	7.924
12	7.289	7.647	7.953	8.302	8.535

k = number of failures observed in incomplete sample

$P[W_{a^\circ} \leq \chi_\alpha] = \alpha$

5. MONOTONE TESTS UNDER IFR ALTERNATIVES

Bickel and Doksum (1968) define a test ϕ to be monotone in the normalized spacings $D_1, \ldots, D_n$ if $\phi\left(D_1', \ldots, D_n'\right) \leq \phi(D_1, \ldots, D_n)$ for all $(D_1, \ldots, D_n)$ and $\left(D_1', \ldots, D_n'\right)$ such that for $i < j$, $D_i' \geq D_j'$ implies $D_i \geq D_j$. We show that if D_i is replaced by $\int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du$ in the incomplete data case, then a monotone test is unbiased for testing H_0 versus H_1 when $n(u) \geq 0$ for $u \geq 0$. The test rejects H_0 for large values of ϕ.

We need

Lemma 3.

Let $r(u)\uparrow$ and $n(u) \geq 0$ for $u \geq 0$. Then for $0 \leq a < b \leq c < d$,

$$\frac{\int_a^b n(u)r(u)du}{\int_a^b n(u)du} \leq \frac{\int_c^d n(u)r(u)du}{\int_c^d n(u)du} .$$

Proof:

$$\frac{\int_a^b n(u)r(u)du}{\int_a^b n(u)du} \leq \frac{r(b)\int_a^b n(u)du}{\int_a^b n(u)du} \leq \frac{r(c)\int_c^d n(u)du}{\int_c^d n(u)du} \leq \frac{\int_c^d n(u)r(u)du}{\int_c^d n(u)du} . \; ||$$

From Lemma 3, we immediately obtain

Theorem 2.

Let ϕ be a monotone test of H_0 versus H_1 based on a sample of incomplete data as described in the introduction. Then

$$E\left[\phi\left(\int_0^{Z_{(1)}} n(u)du, \ldots, \int_{Z_{(k-1)}}^{Z_{(k)}} n(u)du\right) \mid F \text{ IFR}\right] \geq$$

$$\geq E[\phi(Y_1, \ldots, Y_k) \mid F \text{ exponential}] ,$$

where $Y_1, \ldots, Y_k$ independent exponentially distributed random variables.

Proof:

For $i < j$,

$$\frac{\int_{Z_{(j-1)}}^{Z_{(j)}} n(u)du}{\int_{Z_{(i-1)}}^{Z_{(i)}} n(u)du} \leq \frac{\int_{Z_{(j-1)}}^{Z_{(j)}} r(u)n(u)du}{\int_{Z_{(i-1)}}^{Z_{(i)}} r(u)n(u)du} \overset{st}{=} \frac{Y_j}{Y_i} .$$

The inequality follows from Lemma 3; the stochastic equality follows from Lemma 1.

Thus $\phi(Y_1, \ldots, Y_k) \overset{st}{\leq} \phi\left(\int_0^{Z_{(1)}} n(u)du, \ldots, \int_{Z_{(k-1)}}^{Z_{(k)}} n(u)du\right)$. The conclusion follows by taking expectations. ||

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